Period 4, April 29, 2025

Geometric sequence	Common ratio
$1, 5, 25, 125, 625, \ldots$	$r = \frac{5}{1} = 5$
$4, 8, 16, 32, 64, \ldots$	$r = \frac{8}{4} = 2$
$6, -12, 24, -48, 96, \ldots$	$r = \frac{-12}{6} = -2$
$9, -3, 1, -\frac{1}{3}, \frac{1}{9}, \ldots$	$r = \frac{-3}{9} = -\frac{1}{3}$

Definition of a Geometric Sequence

A **geometric sequence** is a sequence in which each term after the first is obtained by multiplying the preceding term by a fixed nonzero constant. The amount by which we multiply each time is called the **common ratio** of the sequence.

r

Write the first six terms of the geometric sequence with first term 6 and common ratio $\frac{1}{3}$.

$6, 6\left(\frac{1}{3}\right), 6\left(\frac{1}{3}\right)^2, 6\left(\frac{1}{3}\right)^3, 6\left(\frac{1}{3}\right)^4, 6\left(\frac{1}{3}\right)^5$ $\qquad a_1 r^{n-1} = a_n$

$6, 2, \frac{6}{9}, \frac{6}{27}, \frac{6}{81}, \frac{6}{243}$

$a_1 = 6$

$r = \frac{1}{3}$

$6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \frac{2}{81}$

$a_n = 6\left(\frac{1}{3}\right)^{n-1}$

General Term of a Geometric Sequence

The nth term (the general term) of a geometric sequence with first term a_1 and common ratio r is

$$a_n = a_1 r^{n-1}.$$

a_8

Find the eighth term of the geometric sequence whose first term is −4 and whose common ratio is −2.

$a_n = a_1(r)^{n-1}$

$a_n = -4(-2)^{n-1}$

$a_8 = -4(-2)^{8-1} = -4 \cdot -2^7 = -4 \cdot -128 = 400 + 112 = 512$

the arithmetic sequence might initially have.

The accompanying table shows the population of the United States in 2010, with estimates for 2011 through 2020.

$2010 - 2009 = 1$

Year	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020
Population (millions)	308.7	310.9	313.0	315.2	317.4	319.7	321.9	324.2	326.4	328.7	331.0

a_1

a. Show that the population is increasing geometrically.
b. Write the general term for the geometric sequence modeling the population of the United States, in millions, n years after 2009.
c. Project the U.S. population, in millions, for the year 2030.

GREAT QUESTION !

Why are estimates given for 2011

$2030 - 2009 = 21$

$r = \frac{310.9}{308.7} = 1.0071267$

$r \approx 1.007$

$r = \frac{313}{310.9} = 1.0067546$

$(315.2)(1.007) = 317.4064$

$(317.4)(1.007) = 319.6218$

$r = \frac{315.2}{313} = 1.0070288$

$r \approx 1.007$ $\quad a_1 = 308.7$

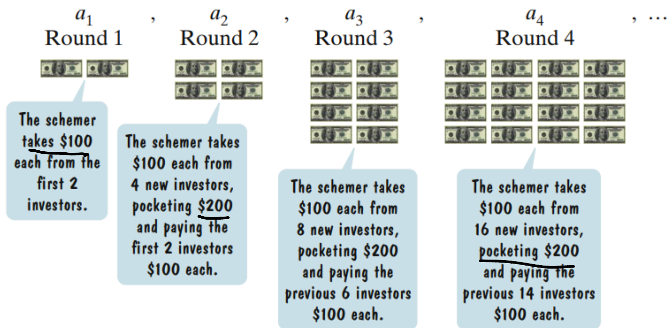
$a_n = 308.7(1.007)^{n-1}$

$a_n = 308.7(1.007)^{21-1}$

$a_n = 308.7(1.007)^{20} = (308.7)(1.149713)$

≈ 354.916 million

A Ponzi scheme is an investment fraud that pays returns to existing investors from funds contributed by new investors rather than from legitimate investment activity. Here's a simplified example:



The number of investors needed to continue this Ponzi scheme,

2, 4, 8, 16, , 32, 64, 128, 256, 512, 1024, 2048

and the money collected in each round,

$200, $400, $800, $1600, 3200, 6 400, 12800, 25600, 51,200, 102400

204,800…

$100 + 200 + 200 + 200$

$\cdot 2 \quad \cdot 2$

$200 + 400 + 800 + 1600 + 3200 + 6400 + 12800 = 25,400$

7 Terms

Geometric Series

$a_1 = 200$

$r = 2$

$n = 7$

$$S_n = \frac{a_1(1 - r^n)}{1 - r},$$

$$\frac{200(1-2^7)}{1-(2)} = \frac{200(1-128)}{-1}$$

$$\frac{200(-127)}{-1} = 25,400$$

The Sum of the First n Terms of a Geometric Sequence

The sum, S_n, of the first n terms of a geometric sequence is given by

$$S_n = \frac{a_1(1 - r^n)}{1 - r},$$

in which a_1 is the first term and r is the common ratio ($r \neq 1$).

Find the sum of the first 18 terms of the geometric sequence: $2, -8, 32, -128, \ldots$.

$r = -4 \quad \cdot -4$

$4^1 = 4$
$4^2 = 16$
$4^3 = 64$
$4^4 = 256$
$4^5 = 1024$

x $\dfrac{2(1-(-4)^{18})}{1-(-4)}$

$a_1 = 2$
$n = 18$

$\dfrac{2(1 - 68719476736)}{5} = \dfrac{2(-68719476735)}{5}$

-27487790694

Find the following sum: $\sum_{i=1}^{10} 6 \cdot 2^i = 6\cdot2^1 + 6\cdot2^2 + 6\cdot2^3 + 6\cdot2^4 + \cdots 6\cdot2^{10}$

$= 12 + 24 + 48 + 96 + \cdots 6144$

$\dfrac{a_1(1-r^n)}{1-r} = \dfrac{12(1-2^{10})}{1-2} = \dfrac{12(1-1024)}{-1}$

$\dfrac{12\cdot(-1023)}{-1} = 12{,}276$

$a_n = a_1(r)^{n-1}$

$a_n = 0.01(2)^{n-1}$

$a_{31} = 0.01(2)^{31-1}$

$= 0.01(1073741824)$

May 1, 2, 3, 4 ⋯ 31

$= 10{,}737{,}418.24$

$0.01 + 0.02 + 0.04 + 0.08 + \cdots$

A union contract specifies that each worker will receive a 5% pay increase each year for the next 30 years. One worker is paid $55,000 the first year. What is this person's total lifetime salary over a 30-year period?

Year 1 2 3 4

$55000 + 55{,}000(1.05) + 55{,}000(1.05)^2 + 55{,}000(1.05)^3 + \cdots \underbrace{55{,}000(1.05)^{29}}_{226{,}387.45}$

$$S_{30} = \frac{55(1-(1.05)^{30})}{1-1.05} = \frac{55(1-4.3219)}{-0.05}$$

$$\frac{55(-3.3219)}{-0.05} = 3654.1366 \cdot 1000$$

$$= 3{,}654{,}136$$

Annuities

The compound interest formula

$$A = P(1 + r)^t$$

gives the future value, A, after t years, when a fixed amount of money, P, the principal, is deposited in an account that pays an annual interest rate r (in decimal form) compounded once a year. However, money is often invested in small amounts at periodic intervals. For example, to save for retirement, you might decide to place $1000 into an Individual Retirement Account (IRA) at the end of each year until you retire. An **annuity** is a sequence of equal payments made at equal time periods. An IRA is an example of an annuity.

Suppose P dollars are deposited into an account at the end of each year. The account pays an annual interest rate, r, compounded annually. At the end of the first year, the account contains P dollars. At the end of the second year, P dollars is deposited again. At the time of this deposit, the first deposit has received interest earned during the second year. The **value of the annuity** is the sum of all deposits made plus all interest paid. Thus, the value of the annuity after two years is

$$P + P(1 + r).$$

Deposit of P dollars at end of second year

First-year deposit of P dollars with interest earned for a year

The value of the annuity after three years is

$$P \quad + \quad P(1 + r) \quad + \quad P(1 + r)^2.$$

Deposit of P dollars at end of third year

Second-year deposit of P dollars with interest earned for a year

First-year deposit of P dollars with interest earned over two years

The value of the annuity after t years is

$$P + P(1 + r) + P(1 + r)^2 + P(1 + r)^3 + \cdots + P(1 + r)^{t-1}.$$

Deposit of P dollars at end of year t

Common Ratio

First-year deposit of P dollars with interest earned over $t - 1$ years

This is the sum of the terms of a geometric sequence with first term P and common ratio $1 + r$. We use the formula

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$

to find the sum of the terms:

$$S_t = \frac{P[1 - (1 + r)^t]}{1 - (1 + r)} = \frac{P[1 - (1 + r)^t]}{-r} = \frac{P[(1 + r)^t - 1]}{r}.$$

Value of an Annuity: Interest Compounded n Times per Year

If P is the deposit made at the end of each compounding period for an annuity at r percent annual interest compounded n times per year, the value, A, of the annuity after t years is

$$A = \frac{P\left[\left(1 + \frac{r}{n}\right)^{nt} - 1\right]}{\frac{r}{n}}.$$

$$\frac{200\left[\left(1+\frac{0.075}{12}\right)^{\overset{480}{12.40}} - 1\right]}{\frac{0.075}{12}} = \frac{200[19.8988 - 1]}{0.00625}$$

$$= 604761.6$$

At age 25, to save for retirement, you decide to deposit $200 at the end of each month into an IRA with a historical return equivalent to 7.5% compounded monthly.

a. Using the historical rate of return, how much will you have from the IRA when you retire at age 65?

b. Find the amount earned on your investment.

How much invested

2400 per year

$40(2400) = 96{,}000$

$P = 200$

$r = 0.075 = 7.5\%$

$T = 40$

$n = 12 =$ compounded monthly

$\approx 604{,}765$

$= \$604{,}765 - \$96{,}000 = \$508{,}765$

$604761 - 96{,}000 = 508{,}761$

An infinite sum of the form

$$a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-1} + \cdots$$

with first term a_1 and common ratio r is called an **infinite geometric series**.

$2 + 4 + 8 + 16 + 32 + 64 + 128 + \cdots = \infty$

$r = 2$ $\quad 2 > 1$

$6 + 2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81} + \cdots$ $\quad S_\infty = \frac{a_1}{1-r}$

$r = \frac{1}{3}$ $\quad$ when $|r| < 1$

$$S_\infty = \frac{6}{1-\frac{1}{3}} = \frac{\frac{6}{1}}{\frac{2}{3}} = \frac{6}{1} \cdot \frac{3}{2} = \frac{18}{2} = 9$$

The Sum of an Infinite Geometric Series

If $-1 < r < 1$ (equivalently, $|r| < 1$), then the sum of the infinite geometric series

$$a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots,$$

in which a_1 is the first term and r is the common ratio, is given by

$$S = \frac{a_1}{1 - r}.$$

If $|r| \geq 1$, the infinite series does not have a sum.

Find the sum of the infinite geometric series: $\frac{3}{8} - \frac{3}{16} + \frac{3}{32} - \frac{3}{64} + \cdots$.

Express $0.\overline{7}$ as a fraction in lowest terms.

$$S_\infty = \frac{\frac{7}{10}}{1 - \frac{1}{10}} = \frac{\frac{7}{10}}{\frac{9}{10}} = \frac{7}{\cancel{10}} \cdot \frac{\cancel{10}}{9}$$

$$= \frac{7}{9}$$

Solution

$$0.\overline{7} = 0.7777\ldots = \frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \frac{7}{10{,}000} + \cdots$$

$r = \frac{1}{10}$

$a_1 = \frac{7}{10}$

EXAMPLE 10 **Tax Rebates and the Multiplier Effect**

A tax rebate that returns a certain amount of money to taxpayers can have a total effect on the economy that is many times this amount. In economics, this phenomenon is called the **multiplier effect**. Suppose, for example, that the government reduces taxes so that each consumer has \$2000 more income. The government assumes that each person will spend 70% of this (= \$1400). The individuals and businesses receiving this \$1400 in turn spend 70% of it (= \$980), creating extra income for other people to spend, and so on. Determine the total amount spent on consumer goods from the initial \$2000 tax rebate.

Solution The total amount spent is given by the infinite geometric series

$$r = 0.7$$

$$1400 + \underbrace{980}_{\text{70\% of 1400}} + \underbrace{686}_{\text{70\% of 980}} + \cdots .$$

$$S_\infty = \frac{1400}{1-0.7} = \frac{1400}{.3} = 4666.67$$

Homework

$$\begin{array}{r} 23097 \\ -21700 \\ \hline 1397 \end{array} \qquad \begin{array}{r} 24494 \\ -23097 \\ \hline 01397 \end{array} \qquad d=1397$$

$$a_n = a_1 + d(n-1) = 21700 + 1397(n-1)$$

In the sequence 21700, 23097, 24494, 25891, ..., which term is 439403?

a_1

The term 439403 is the $\boxed{\cancel{314}}$ th term.

$$a_n = 21700 + 1397(n-1)$$

$$439403 = 21700 + 1397(n-1)$$

$$-21,700 \qquad -21,700$$

$$\frac{417703}{1397} = \frac{1397(n-1)}{1397}$$

$$299 = n-1$$

$$+1 \qquad +1$$

$$300 = n$$

Write out the first three terms and the last term. Then use the formula for the sum of the first n terms of an arithmetic sequence to find the indicated sum.

$$\sum_{i=1}^{30}(-2i+1) = (-2(1)+1) + (-2(2)+1) + (-2(3)+1) + \ldots (-2(28)+1) + (-2(29)+1) + (-2(30)+1)$$

$$-1 + -3 + -5 + \ldots -55 + -57 + -59$$

Find the first three terms and the last term.

$$-1 + -59 = -60$$

$$\sum_{i=1}^{30}(-2i+1) = (-1) + (-3) + (-5) + \cdots + (-59)$$

$$-3 + -57 = -60$$

Find the sum of the sequence.

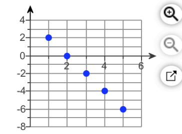
$$\sum_{i=1}^{30}(-2i+1) = -900$$

$$\frac{(a_1+a_n)n}{2} = \frac{(-60)\cdot 30}{2} = -900$$

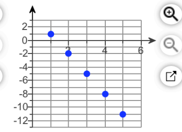
$$-60 \cdot 15 = -900$$

Use the graphs of the arithmetic sequences $\{a_n\}$ and $\{b_n\}$ to find $a_{13} + b_{15}$.

$\{a_n\}$



$\{b_n\}$



$a_{13} + b_{15} = -63$

$-22 + -41 = -63$

a_{13}

$2, 0, -2, -4$

$-2 \quad -2 \quad -2 = d$

$a_n = 2 + -2(n-1)$

$a_n = 2 - 2n + 2$

$a_n = 4 - 2n$

$a_{13} = 4 - 2(13) = 4 - 26$

$= -22$

b_{15}

$1, -2, -5$

$-3 \quad -3 = d$

$b_n = 1 - 3(n-1)$

$b_n = 1 - 3n + 3 = 4 - 3n$

$b_{15} = 4 - 3 \cdot 15 = 4 - 45$

$b_{15} = -41$

Use a system of two equations in two variables, a_1 and d, to write a formula for the general term (the nth term) of the arithmetic sequence whose seventh term, a_7, is 10 and whose ninth term, a_9, is 14.

$a_n = 2n - 4$

$a_7 = 10 \qquad a_9 = 14$

a_7, a_8, a_9

$10, 12, 14 = 10 + 2d$

$+d \quad +d$

$d = 2$

$a_n = a_1 + d(n-1)$

$a_n = a_1 + 2(n-1)$

$a_7 = a_1 + 2(7-1)$

$10 = a_1 + 2 \cdot 6$

$10 = a_1 + 12$

$-12 \qquad -12$

$-2 = a_1$

$a_n = -2 + 2(n-1)$

$= -2 + 2n - 2$

$a_n = 2n - 4$

You are considering two job offers. Company A will start you at $32,000 a year and guarantee a raise of $2500 per year. Company B will start you at a higher salary, $41,000 a year, but will only guarantee a raise of $600 per year. Find the total salary that each company will pay over a ten-year period. Which company pays the greater total amount?

Company A will pay $ 432500 over a ten-year period.

Company B will pay $ 437000 over a ten-year period.

Company B pays the greater total amount.

B

$a_1 = 41{,}000$

$a_2 = 41{,}000 + 600 = 41{,}600$

$a_3 = 42{,}200$

$d = +600$

$a_n = 41{,}000 + 600(n-1)$

$a_n = 41{,}000 + 600n - 600$

$a_n = 40{,}400 + 600n$

$a_{10} = 46{,}400$

$S_{1-10} = \frac{(41{,}000 + 46{,}400)\cdot 10}{2}$

$87400 \cdot 5 = 437{,}000$

Company A

$32{,}000 + [32{,}000 + 2500] + [32{,}000 + 5000]$

$d = 2500$

$a_1 = 32000$

$a_n = 32000 + 2500(n-1)$

$a_1 = 32000$

$a_2 = 34{,}500$

$a_3 = 37{,}000$

$a_9 = 52{,}000$

$a_{10} = 54{,}500$

86 500

86,500

$\sum_{i=1}^{10} [32000 + 2500(n-1)] = 86500 \cdot 5$

$= 432{,}500$

Company A pays $24,000 yearly with raises of $1,400 per year. Company B pays $27,000 yearly with raises of $800 per year. Which company will pay more in year 10? How much more?

Which company pays more in year 10?

- [x] Company A pays more.
- [ ] Both companies pay the same.
- [ ] Company B pays more.
- [ ] There is not enough information.

How much more?

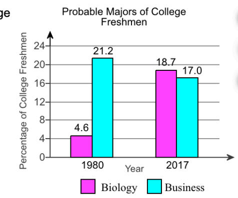
$ 2400

Same Type of Question

The bar graph shows two probable majors of college freshman in 1980 and 2017. In 1980, 4.6% of college freshman declared biology as their probable major. On average, this percentage has increased by approximately 0.4 each year.

a. Write a formula for the nth term of the arithmetic sequence that models the percentage of college freshman who declared biology as their probable major n years after 1979.

b. Use the formula in part (a) to project the percentage of college freshman who will declare biology as their major by 2024.



$d = 0.4$

$a_1 = 4.6 = 1980$ year

$4.6 - .4 = 4.2 = 1979$

$a_1 = 4.2$

$d = 0.4$

1980 is year 1

$1980 - 1979 = 1$

$a_1 = 4.6$

$a_n = 4.6 + 0.4(n-1) = 4.2 + 0.4n$

a. The formula for the nth term of the arithmetic sequence is a_n = 0.4n + 4.2.

(Simplify your answer. Use integers or decimals for any numbers in the expression.)

b. Approximately 22.2 % of college freshmen will declare biology as their probable major by 2024.

(Simplify your answer. Type an integer or a decimal.)

$$\begin{array}{r} 2024 \\ -1979 \\ \hline 45 \end{array}$$

Year $2024 \Rightarrow n = 45$

$a_{45} = 4.2 + 0.4(45)$

$4.2 + 18$

$a_{45} = 22.2$